

A Preliminary Report on Search for Good Examples of Hall's Conjecture.

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1 Introduction.

Consider the equation

$$x^3 - y^2 = k \tag{*}$$

where $x, y \in \mathbb{N}$ and $k \in \mathbb{Z}$. It is easy to see that (*) has infinitely many solutions where $k = 0$ (let $x = t^2$ and $y = t^3$ where t is a natural number). It turns out that (*) only has finitely many solutions in x and y when k is a given integer different from 0. Moreover, it is hard to find solutions of (*) where k is small compared to x and y . Hall's Conjecture states that there exists a constant C such that for any solution of (*) where $k \neq 0$, we have $C\sqrt{x} < |k|$. For more on Hall's Conjecture, see [1] and [3].

Hall's Conjecture is neither proved nor disproved. To shed some light upon the conjecture, researchers has tried to find solutions of (*) where $0 < |k| < \sqrt{x}$. We will refer to such solutions as *good examples of Hall's Conjecture*, and we will say that (x, y, k) is a *good triplet* when $x, y \in \mathbb{N}$ and $0 < |x^3 - y^2| = |k| < \sqrt{x}$.

This paper is a preliminary report on our search for new good examples of Hall's Conjecture. We present a new algorithm that will detect all good examples

within a given search space. We have implemented the algorithm, and our executions have so far found five new good examples.

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2 The Basis for the Algorithm

In this section we will state some basic definitions and theorems. In the next section we will explain our algorithm.

We use capital Latin letters to denote polynomials, and we use small Latin letters to denote numbers.

Definition 2.1 *The polynomials B , C , F and H are defined by*

$$B(q, p, x) = p^2 - q^2x \quad (1)$$

$$C(q, p, x, y) = p^3 - 3pq^2x + 2q^3y \quad (2)$$

$$F(q, p, x, y) = 4pC - 3B^2 \quad (3)$$

$$H(q, p, x, y) = 9FB - 8C^2 \quad (4)$$

Lemma 2.2 *We have*

$$F = p^4 - (6p^2x + 8pqy - 3q^2x^2)q^2$$

and

$$H = p^6 - (15p^4x + 40p^3qy - 45p^2q^2x^2 + 24pq^3xy + 27q^4x^3 - 32q^4y^2)q^2$$

Proof: The lemma follows straightforwardly from the definition of the polynomials F and H . **QED.**

Theorem 2.3 *We have*

$$(1) \ C \equiv p^3 \pmod{q^2}$$

$$(2) \ F \equiv p^4 \pmod{q^2}$$

$$(3) \ H \equiv p^6 \pmod{q^2}$$

$$(4) \ H \equiv -8C^2 \pmod{9|F|}.$$

$$(5) \ p^4 - 2pC + F \equiv 0 \pmod{q^3}$$

$$(6) \ 4p^6 - 5p^3C + H \equiv 0 \pmod{q^3}.$$

Proof: Clause (1) and (4) follow straightforwardly from the definition of the polynomials B , C and H . Clause (2) and (3) hold by Lemma 2.2. Furthermore, (5) holds since

$$\begin{aligned}
p^4 - 2pC + F &\stackrel{(a)}{=} p^4 - 2pC + 4pC - 3B^2 \\
&= p^4 + 2pC - 3B^2 \\
&\stackrel{(b)}{=} p^4 + 2pC - 3(p^2 - q^2x)^2 \\
&= p^4 + 2pC - 3p^4 + 6p^2q^2x - 3q^4x^2 \\
&= -2p^4 + 2pC + 6p^2q^2x - 3q^4x^2 \\
&\stackrel{(c)}{=} -2p^4 + 2p(p^3 - 3pq^2x + 2q^3y) + 6p^2q^2x - 3q^4x^2 \\
&= -2p^4 + 2p^4 - 6p^2q^2x + 4pq^3y + 6p^2q^2x - 3q^4x^2 \\
&= 4pq^3y - 3q^4x^2 \\
&= (4py - 3qx^2)q^3
\end{aligned}$$

where the equalities labeled (a), (b) and (c) hold by the definitions of respectively F , B and C . The proof of (6) is also straightforward. **QED.**

The proof of the next theorem is long and involved. Most of the poof can be found in the Section 5. The reader should note that the p and the q given by the theorem are such that $\frac{p}{q}$ is a rational approximation to \sqrt{x} .

Theorem 2.4 *Let (x, y, k) be a good triplet. Then, there exists $p, q \in \mathbb{N}$ such that $p < x^{2/3} + 1$ and $q < x^{1/6}$ and*

- $0 < C(q, p, x, y) < 3qx^{1/6} + 1$
- $|F(q, p, x, y)| < 8q + 1$
- $|H(q, p, x, y)| < 72q^4 + 1$.

The final theorem in this section is a straightforward consequence of Definition 2.1.

Theorem 2.5 *We have*

- $B = \frac{H + 8C^2}{9F}$
- $p = \frac{F + 3B^2}{4C}$
- $x = \frac{p^2 - B}{q^2}$
- $y = \frac{3pq^2x - p^3 + C}{2q^3}$.

3 The algorithm.

Our algorithm works by examining quadruples (q, f, c, h) . For every good triplet (x, y, k) , Theorem 2.4 guarantees at least one quadruple (q, f, c, h) such that

- (i) $q < x^{1/6}$
- (ii) $0 < c = C(q, p, x, y) < 3qx^{1/6} + 1$
- (iii) $f = |F(q, p, x, y)| < 8q + 1$
- (iv) $h = |H(q, p, x, y)| < 72q^4 + 1$.

The algorithm uses the equalities in Theorem 2.5 to compute good triplets from quadruples. The algorithm uses the modulo equivalences of Theorem 2.3 to find quadruples that may yield a good triplets.

Choose x_{\max} and set $q_{\max} = x_{\max}^{1/6}$. The following algorithm finds all good triplets with $x < x_{\max}$:

1. For $q = 2, \dots, q_{\max}$, examine the corresponding values of f, c, h as outlined in the following steps:
2. Introduce the auxiliary variable p_0 . Examine each $p_0 < q$ such that p_0 and q are co-prime. Let p be such that $p_0 \equiv p \pmod{q}$ (we do not compute p). By Theorem 2.3, $f \equiv p^4 \pmod{q^2}$, so $f \equiv p_0^4 \pmod{q}$. Also, by (iii), $|f| < 8q + 1$. Thus we can describe (and compute) the set S_f of possible values of f by

$$S_f = \{iq + (p_0^4 \pmod{q}) \mid -8 \leq i \leq 8\}.$$

3. For $f \in S_f$, introduce the auxiliary variable p_1 . Examine all p_1 such that $p_1^4 \equiv f \pmod{q^2}$ and $0 < p_1 < q^2$. Define the set

$$S = \{(f \pmod{q^2}) + qi \mid 0 \leq i \leq q - 1\}.$$

Then $p_1 \in S$; we find the admissible values for p_1 by checking the elements of S . There are at most 4 possible values of p_1 . Now $p_1 \equiv p \pmod{q^2}$.

4. By Theorem 2.3, $c \equiv p^3 \pmod{q^2}$. An upper bound for c is provided by (ii). Hence, the possible values of c are

$$(p_1^3 \pmod{q^2}), (p_1^3 \pmod{q^2}) + q^2, \dots, (p_1^3 \pmod{q^2}) + q^2 \left\lfloor \frac{q_{\max}}{q} \right\rfloor.$$

5. Introduce the auxiliary variable p_2 . Examine all p_2 such that $p_2 \equiv p_1 \pmod{q^2}$ and $p_2^4 - 2p_2c + f \equiv 0 \pmod{q^3}$ and $p_2 < q^3$. These p_2 satisfies clause (6) in Theorem 2.3.
6. Introduce $h_2 \equiv 5cp_2^3 - 4p_2^6 \pmod{q^3}$. This implies $h_2 \equiv h \pmod{q^3}$.
7. Introduce $h_3 \equiv h_2 \pmod{q^3}$ such that h_3 satisfies (4) and (6) in Theorem 2.3. This means that we must find h_3 such that $h_3 \equiv h_2 \pmod{q^3}$ and $h_3 \equiv (-8c^2) \pmod{9f}$. By the Chinese Remainder Theorem:

if $q \nmid 3$ then

$$h_3 = h_2(9|f|)\text{inv}_{q^3}(9|f|) + (-8c^2)q^3\text{inv}_{(9|f|)}(q^3)$$

$$h \equiv h_3 \pmod{9q^3|f|}$$

if $q \mid 3$ then

$$h_3 = h_2(|f|\text{inv}_{q^3}(|f|) + (-8c^2)q^3\text{inv}_{|f|}(q^3))$$

$$h \equiv h_3 \pmod{q^3|f|}$$
8. Having established the preceding relation for h , using that $0 < |h| < 72q^4$, we are now in a position to find the possible values of h (given q, f, c).
9. The final steps of the algorithm consists of using the equations in Theorem 2.5 to compute a good triplet from q, f, c, h (if such a triplet exists):
10. $b = \frac{h+8c^2}{9f}$; if b is not an integer, then we do not have a good triplet.
11. $p = \frac{f+3b^2}{4c}$; if p is not an integer, then we do not have a good triplet.
12. $x = \frac{p^2-b}{q^2}$; if x is not an integer, then we do not have a good triplet.
13. $y = \frac{3pq^2x-c}{2q^3}$; if y is not an integer, then we do not have a good triplet.
14. $k = x^3 - y^2$; if $|k| < \sqrt{x}$, we have a good triplet.

Validity of the algorithm

The algorithm works by identifying rational approximations $\frac{p}{q}$ to \sqrt{x} , where x is the first component of a good triplet. These approximations are found by examining tuples (q, f, c, h) : for every good triplet (x, y, k) at least one such tuple T_x exists, with $q < x^{1/6}$ (Theorem 2.4). Now assume that $\frac{p}{q}$ is an approximation to \sqrt{x} where x yields a “good example”. Also assume that $x < x_{\max}$. By the following argument, the algorithm exhausts all the possibilities for (q, f, c, h) .

For each q , we examine all p_0 such that $1 \leq p_0 < q$, with p_0 and q are co-prime (if p_0 and q should have a common factor r , then $\frac{p_0}{q}$ would be equivalent to $\frac{p_0/r}{q/r}$).

By Theorem 2.3 $f \equiv p^4 \pmod{q^2}$. Writing $p = rq + p_0$, we get $f \equiv p^4 \equiv 4rqp_0^3 + p_0^4 \pmod{q^2}$. By Theorem 2.4, $|f| < 8q + 1$. Thus $f = (p_0^4 \pmod{q^2}) + sq$, with $|s| \leq 8$ (step 2 in the preceding description of the algorithm).

By Theorem 2.3, $f \equiv p^4 \pmod{q^2}$. Let S_{p_1} be the set of (at most 4) solutions of the quadratic congruence $f \equiv p_1^4 \pmod{q^2}$. Then $(p \pmod{q^2}) \in S_{p_1}$.

Now, for each p_1 in S_{p_1} , let S_{c,p_1} be the set consisting of the values

$$(p_1^3 \pmod{q^2}), (p_1^3 \pmod{q^2}) + q^2, \dots, (p_1^3 \pmod{q^2}) + q^2 \left\lfloor \frac{q_{\max}}{q} \right\rfloor.$$

The c corresponding to the tuple T_x will be in S_{c,p_1} for one of the p_1 in S_{p_1} .

Let $p_2 \equiv p \pmod{q^3}$. By clause (5) in Theorem 2.3, we have $p_2^4 - 2p_2c + f \equiv 0 \pmod{q^3}$. Define the set $S_{p_2,c,f}$ = the elements of the series $p_1, p_1 + q^2, \dots$ (up to q^3) that satisfy this clause. Then $p_2 \in S_{p_2,c,f}$, with c and f the corresponding values in T_x .

By clauses (4) and (6) in Theorem 2.3, we have $h \equiv -8c^2 \pmod{(9|F|)}$ and $h \equiv 5cp^3 - 4p^6 \pmod{q^3}$. Introduce the variable h_3 = the smallest positive integer that satisfies these clauses; and define the set $S_h(c, f)$ as the collection of all values $< 72q^4 + 1$ that satisfies these clauses (for given p_0, c, f). Then $h \in S_h(c, f)$, and the algorithm will accordingly produce the output of the quantities b, p, x, y, k .

4 Results.

In order to investigate the feasibility of the algorithm, the algorithm was implemented in Python and tested with values of q_{\max} up to 1000. As results looked promising, the algorithm was reimplemented in C, using the Gnu Multi-Precision library to carry out operations with arbitrary-length integers. This program was run with $q_{\max} = 10000$ (corresponding to a x_{\max} of 10^{24}). This run took 57 processor-days; after 35 days it produced the solution #44 in Table 1. A subsequent run, with $q_{\max} = 20000$ (corresponding to a x_{\max} of 64×10^{24}), took about 441 processor-days and reproduced a solution earlier found by Calvo et al. [6]. Currently the program is running on the Norwegian national computing facilities (Notur). So far, five new good examples have been found. All known good examples are included in Table 1.

#	x	$r^{(10)}$	$\frac{p}{q}$	Comments
1	2	1.42		1)
2	5234	4.26	$\frac{217}{3}$	2), 3)
3	8158	3.76	$\frac{271}{3}$	2), 3)
4	93844	1.03	$\frac{919}{3}$	2), 3), 9)
5	367806	2.93	$\frac{1213}{2}$	2), 3)
6	421351	1.05	$\frac{5193}{8}$	2), 3)
7	720114	3.77	$\frac{4243}{8}$	2), 3)
8	939787	3.16	$\frac{6786}{7}$	2), 3)
9	28187351	4.87	$\frac{90256}{17}$	2), 3)
10	110781386	1.23	$\frac{115778}{11}$	2), 3)
11	154319269	1.08	$\frac{211183}{17}$	2), 3)
12	384242766	1.34	$\frac{176419}{9}$	2), 3)
13	390620082	1.33	$\frac{177877}{9}$	2), 3)
14	3790689201	2.20	$\frac{430980}{7}$	3)
15	65589428378	2.19	$\frac{768313}{3}$	4)
16	952764389446	1.15	$\frac{79063817}{81}$	4)
17	12438517260105	1.27	$\frac{507863263}{144}$	4)
18	35495694227489	1.15	$\frac{1030703950}{173}$	4)
19	53197086958290	1.66	$\frac{437617999}{60}$	4)
20	5853886516781223	46.60	$\frac{6426898417}{84}$	4)
21	12813608766102806	1.30	$\frac{17319173410}{153}$	4)
22	23415546067124892	1.46	$\frac{68094518942}{445}$	4)
23	38115991067861271	6.50	$\frac{108354409918}{555}$	4)
24	322001299796379844	1.04	$\frac{387001980055}{682}$	4) 9)
25	471477085999389882	1.38	$\frac{83083668769}{121}$	4)
26	810574762403977064	4.66	$\frac{359227383073}{399}$	4)
27	9870884617163518770	1.90	$\frac{4524186815567}{1440}$	5)
28	42532374580189966073	3.47	$\frac{8386886845023}{1286}$	5)
29	44648329463517920535	1.79	$\frac{4603857036361}{689}$	5)
30	51698891432429706382	1.75	$\frac{9318491574937}{1296}$	5)
31	231411667627225650649	3.71	$\frac{14649368819024}{963}$	5)
32	601724682280310364065	1.88	$\frac{39714194816596}{1619}$	5)
33	4996798823245299750533	2.17	$\frac{250164969159375}{3539}$	5)
34	5592930378182848874404	1.38	$\frac{32531865160357}{435}$	5)
35	14038790674256691230847	1.27	$\frac{392068197831386}{3309}$	5)
36	77148032713960680268604	10.18	$\frac{633004435512983}{2279}$	6)
37	180179004295105849668818	5.65	$\frac{678311009850201}{1598}$	6)
38	372193377967238474960883	1.33	$\frac{539307656512279}{884}$	5)
39	664947779818324205678136	16.53	$\frac{3652370552518775}{4479}$	5)
40	2028871373185892500636155	1.14	$\frac{11181418791644809}{7850}$	6)
41	10747835083471081268825856	1.35	$\frac{42884607802081920}{13081}$	7)
42	37223900078734215181946587	1.87	$\frac{46777434586297319}{7667}$	5)
43	69586951610485633367491417	1.22	$\frac{72198966044283893}{8655}$	8)
44	3690445383173227306376634720	1.51	$\frac{121619570207840431}{2002}$	5)
45	162921297743817207342396140787	10.65	$\frac{20237053244197156774}{50137}$	8)
46	1114592308630995805123571151844	1.04	$\frac{95524640670266092418}{90481}$	9)
47	39739590925054773507790363346813	3.75	$\frac{211515916260522809737}{33553}$	8)
48	862611143810724763613366116643858	1.10	$\frac{930889835660831460142}{31695}$	8)
49	1062521751024771376590062279975859	1.01	$\frac{1095269810850785984986}{33601}$	8)
50	6078673043126084065007902175846955	1.03	$\frac{20224028423712303104623}{259396}$	5)

Table 1: See Table 2 for comments.

- 1) This solution is not found by the algorithm presented here.
- 2) Found by M.Hall [3].
- 3) Found by Gebel, Pethö and Zimmer [4].
- 4) Found by N.D.Elkies [2].
- 5) Found by Jiménez Calvo, Herranz and Sáez [1].
- 6) Found by Johan Bosman utilizing the software of
Jiménez Calvo, Herranz and Sáez [1].
- 7) Found by Jiménez Calvo [6].
- 8) Found by the authors of this paper.
- 9) From the Danilov-Elkies infinite Fermat-Pell family.
- 10) $r = k/\sqrt{x}$. High values of r indicate that Hall's Conjecture is false.

Table 2: Comments to Table 1.

5 The Proof of Theorem 2.4

Lemma 5.1 *Let (x, y, k) be a good triplet. Then, there exists $\gamma \in \mathbb{R}$ such that*

$$y = x^{3/2}(1 + \gamma) \quad \text{and} \quad \frac{-|k|}{2x^{5/2}} < \gamma < \frac{|k|}{2x^{6/2}}.$$

Proof: For any $x, y \in \mathbb{N}$, we have $\gamma \in \mathbb{R}$ such that $y = x^{3/2}(1 + \gamma)$. For convenience, let w denote \sqrt{x} . Then, we have

$$y = x^{3/2}(1 + \gamma) = w^3(1 + \gamma). \quad (*)$$

Furthermore, we have,

$$\begin{aligned}
\gamma w^3 &= w^3(1 + \gamma) - w^3 \\
&= y - w^3 & (*) \\
&= (y^2 - w^6)/(y + w^3) \\
&= (y^2 - x^3)/(y + w^3) & \text{since } w = \sqrt{x} \\
&= -k/(y + w^3) & \text{since } x^3 - y^2 = k
\end{aligned}$$

This establishes that $\gamma w^3 = -k/(y + w^3)$, and thus

$$\gamma = \frac{-k}{(y + w^3)w^3} = \frac{-k}{yw^3 + w^6}. \quad (**)$$

Next, note that y cannot equal w^3 (if $y = w^3 = x^{3/2}$, then (x, y, k) will not be a good triplet as $x^3 - y^2 = 0$). So, we have either $y > w^3$ or $y < w^3$.

Assume that $y > w^3 = x^{3/2}$. Then, since $x^3 - y^2 = k$, we have $k < 0$. By (**), we have

$$0 < \gamma = \frac{-k}{yw^3 + w^6} < \frac{-k}{2w^6} = \frac{|k|}{2x^{\frac{6}{2}}}.$$

Assume that $y < w^3 = x^{3/2}$. Then, since $x^3 - y^2 = k$, we have $k > 0$. Moreover, we have $w^2 < y$ (if $y \leq w^2 = x$, then (x, y, k) will not be a good triplet as $x^3 - y^2 > \sqrt{x}$). Now, by (**), we have

$$0 > \gamma = \frac{-k}{yw^3 + w^6} > \frac{-k}{w^5 + w^6} > \frac{-k}{2w^5} = \frac{-|k|}{2x^{5/2}}.$$

QED.

Lemma 5.2 *Let (x, y, k) be a good triplet. Then, there exist $p, q, Q \in \mathbb{N}$ and $\delta \in \mathbb{R}$ such that (i) $p = q\sqrt{x}(1 + \delta)$, (ii) $x^{1/18} < q < x^{1/6} < Q$ and*

$$(iii) \quad \frac{1}{q\sqrt{x}(Q + q)} < |\delta| < \frac{1}{q\sqrt{x}Q}.$$

Proof: First we note that \sqrt{x} is an irrational number when (x, y, k) is a good triplet. (If \sqrt{x} is a natural number, then (x, y, k) will not be a good triplet as $k = 0$. But \sqrt{x} is either a natural number or an irrational number. Thus we conclude that \sqrt{x} is irrational.)

Let a_0, a_1, a_2, \dots be the coefficients for the simple continued fraction for \sqrt{x} , that is

$$\sqrt{x} = \lim_{n \rightarrow \infty} [a_0; a_1, \dots, a_n]$$

and let h_i and k_i be, respectively, the nominator and the denominator of the convergent $[a_0; a_1, \dots, a_i]$, that is $\frac{h_i}{k_i} = [a_0; a_1, \dots, a_i]$. Then, for any $i \in \mathbb{N}$, we have

$$\frac{1}{k_i(k_i + k_{i+1})} < \left| \frac{h_i}{k_i} - \sqrt{x} \right| < \frac{1}{k_i k_{i+1}}$$

and $k_i < k_{i+1}$. For more on continued fractions, see e.g. [5]. Now, pick the least j such that $k_{j+1} > x^{1/6}$. Let $q = k_j$, let $p = h_j$ and let $Q = k_{j+1}$. Then, we have

$$\frac{1}{q(q + Q)} < \left| \frac{p}{q} - \sqrt{x} \right| < \frac{1}{qQ}$$

where $q < x^{1/6} < Q$ (we cannot have $q = x^{1/6}$ as $x^{1/6} \notin \mathbb{N}$). Next, let δ be the real number such that $p = q\sqrt{x}(1 + \delta)$. Then, we have

$$\frac{1}{q(q + Q)} < \left| \frac{q\sqrt{x}(1 + \delta)}{q} - \sqrt{x} \right| < \frac{1}{qQ}.$$

Thus

$$\frac{1}{q(q+Q)} < |\sqrt{x}\delta| < \frac{1}{qQ}.$$

Thus

$$\frac{1}{q\sqrt{x}(q+Q)} < |\delta| < \frac{1}{q\sqrt{x}Q}.$$

QED.

The next proposition corresponds to the first clause of Theorem 2.4.

Proposition 5.3 *Let (x, y, k) be a good triplet. Then, there exist $p, q \in \mathbb{N}$ such that*

$$0 < C(q, p, x, y) < 3qx^{1/6} + 1.$$

Moreover, $q < x^{1/6}$ and $p < x^{2/3} + 1$.

Proof: To improve the readability, we will use w to denote \sqrt{x} . First we observe that the two preceding lemmas yield $p, q \in \mathbb{N}$ and $\gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} C &= p^3 - 3pq^2w^2 + 2q^3y && \text{def. of } C \\ &= p^3 - 3pq^2w^2 + 2q^3[w^3(1+\gamma)] && \text{Lem 5.1} \\ &= [qw(1+\delta)]^3 - 3[qw(1+\delta)]q^2w^2 + 2q^3[w^3(1+\gamma)] && \text{Lem 5.2} \\ &= q^3w^3(1+\delta)^3 - 3q^3w^3(1+\delta) + 2q^3w^3(1+\gamma) \\ &= q^3w^3[(1+\delta)^3 - 3(1+\delta) + 2(1+\gamma)] \\ &= q^3w^3(\delta^3 + 3\delta^2 + 2\gamma). \end{aligned}$$

Thus, whenever (x, y, k) is a good triplet, we can fix $p, q \in \mathbb{N}$ and $\gamma, \delta \in \mathbb{R}$ such that $C(q, p, x, y) = q^3w^3(\delta^3 + 3\delta^2 + 2\gamma)$. Moreover, Lemma 5.2 states that $q < x^{1/6}$. We invite the reader to check that it follows from Lemma 5.2 that $p < x^{2/3} + 1$. Next, we will use the bounds given in Lemma 5.1 and Lemma 5.2, to prove that

$$0 < q^3w^3(\delta^3 + 3\delta^2 + 2\gamma) = C \tag{*}$$

and

$$C = q^3w^3(\delta^3 + 3\delta^2 + 2\gamma) < 3qw^{1/3} + 1 \tag{**}$$

This, will complete the proof of the proposition (since $w^{1/3} = x^{1/6}$).

We prove (*). It follows from Lemma 5.2 (iii) that $|\delta| < \frac{1}{2}$ and, thus, we have

$$\delta^3 + 3\delta^2 = 2\delta^2 + \delta^2(1+\delta) > 0. \tag{\dagger}$$

Hence

$$\begin{aligned}
C &= q^3 w^3 (\delta^3 + 3\delta^2 + 2\gamma) > q^3 w^3 2\gamma & (\dagger) \\
&> q^3 w^3 2 \frac{-|k|}{2w^5} & \text{Lemma 5.1} \\
&= q^3 \frac{-|k|}{w^2} \\
&> \frac{-|k|}{w} & \text{since } q^3 < (x^{1/3})^3 = w \\
&> -1. & \text{since } |k| < w
\end{aligned}$$

(We have $|k| < w$ since $w = \sqrt{x}$ and (x, y, k) is a good triplet.) Now we have proved $C > -1$, but C cannot be 0 as we e.g. have $p = (F + 3B^2)/4C$ (see Theorem 2.5). Thus we conclude that $C > 0$. This proves (*).

We turn to the proof of (**). By Lemma 5.2, there exists $Q > x^{1/6} = w^{1/3}$ such that

$$\begin{aligned}
C &= q^3 w^3 (\delta^3 + 3\delta^2 + 2\gamma) \\
&\leq q^3 w^3 (|\delta|^3 + 3|\delta|^2 + 2|\gamma|) \\
&< q^3 w^3 \left(|\delta|^3 + 3|\delta|^2 + \frac{|k|}{w^6} \right) & \text{Lemma 5.1} \\
&< q^3 w^3 \left(\left[\frac{1}{qwQ} \right]^3 + 3 \left[\frac{1}{qwQ} \right]^2 + \frac{|k|}{w^6} \right) & \text{Lemma 5.2} \\
&= \frac{1}{Q^3} + \frac{3qw}{Q^2} + \frac{q^3 |k|}{w^3} \\
&< \frac{1}{(w^{1/3})^3} + \frac{3qw}{(w^{1/3})^2} + \frac{(w^{1/3})^3 |k|}{w^3} & \text{since } q < w^{1/3} < Q \\
&= \frac{1}{w} + 3qw^{1/3} + \frac{|k|}{w^2} \\
&= 3qw^{1/3} + \frac{w + |k|}{w^2} \\
&= 3qw^{1/3} + \frac{2w}{w^2} & \text{since } k < \sqrt{x} = w \\
&\leq 3qw^{1/3} + 1. & \text{since } w \geq 2
\end{aligned}$$

This completes our proof. **QED.**

The next two proposition correspond to, respectively, the second and third clause of Theorem 2.4. Detailed proofs of these two propositions will not be included in this preliminary report.

Proposition 5.4 *Let (x, y, k) be a good triplet. Then, there exist $p, q \in \mathbb{N}$ such that*

$$|F(q, p, x, y)| < 8q + 1.$$

Moreover, $q < x^{1/6}$ and $p < x^{2/3} + 1$.

Proof: Use Lemma 5.1 and Lemma 5.2 to prove that there exist $p, q \in \mathbb{N}$ and $\gamma, \delta \in \mathbb{R}$ such that

$$F(q, p, x, y) = q^4 x^2 (8\gamma + 8\gamma\delta + 4\delta^3 + \delta^4).$$

Then, use the bounds given in the two lemmas to prove that the proposition holds. **QED.**

Proposition 5.5 *Let (x, y, k) be a good triplet. Then, there exist $p, q \in \mathbb{N}$ such that*

$$|H(q, p, x, y)| < 72q^4 + 1.$$

Moreover, $q < x^{1/6}$ and $p < x^{2/3} + 1$.

Proof: Use Lemma 5.1 and Lemma 5.2 to prove that there exist $p, q \in \mathbb{N}$ and $\gamma, \delta \in \mathbb{R}$ such that

$$H(q, p, x, y) = q^6 x^3 (144\delta\gamma - 32\gamma^2 + 40\delta^3\gamma + 120\delta^2\gamma + 6\delta^5 + \delta^6).$$

Then, use the bounds given in the two lemmas to prove that the proposition holds. **QED.**

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